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LETTER TO THE EDITOR

Supersymmetric classical mechanics

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Abstract. We study the classical properties of a supersymmetric system which is often used as a model for supersymmetric quantum mechanics. It is found that the classical dynamics of the bosonic as well as the fermionic degrees of freedom is fully described by a so-called quasi-classical solution. We also comment on the importance of this quasi-classical solution in the semi-classical treatment of the supersymmetric quantum model.

In 1976, Nicolai [1] introduced supersymmetric quantum mechanics as an example for the occurrence of supersymmetry (SUSY) in non-relativistic quantum mechanics. Independently, in 1981, Witten [2] also suggested SUSY quantum mechanics as a simplified model for the study of the spontaneous SUSY breaking mechanism. The model which has been studied [1–3] is characterized by the following Lagrangian

$$L := \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}V^2(x) + \tfrac{\mathrm{i}}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - V'(x)\bar{\psi}\psi. \tag{1}$$

In the above, x denotes a bosonic degree of freedom and, hence, is an even Grassmann number. In contrast to this, ψ and $\bar{\psi}$ denote fermionic degrees of freedom and, therefore, are odd Grassmann numbers, which means that $\{\psi, \bar{\psi}\} = 0$ and $\psi^2 = 0 = \bar{\psi}^2$. The real-valued function V is the so-called superpotential. The Lagrangian (1) describes the supersymmetrized version of a $(0+1)$-dimensional field theory. In other words, it stems from a supersymmetric field theory formulated in a superspace spanned by the time variable t and two Grassmann variables ε and $\bar{\varepsilon}$ [1, 4, 5]. As a consequence, the dynamical system defined by (1) is invariant under the SUSY transformations

$$\begin{aligned}
\delta x(t) &= \varepsilon\psi(t) + \bar{\psi}(t)\bar{\varepsilon} \\
\delta\psi(t) &= -(\mathrm{i}\dot{x} + V(x))\bar{\varepsilon} \\
\delta\bar{\psi}(t) &= (\mathrm{i}\dot{x} - V(x))\varepsilon.
\end{aligned} \tag{2}$$

The invariance of the system characterized by (1) under the SUSY transformations (2) is obvious as

$$\delta L = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((\dot{x} - \mathrm{i}V)\varepsilon\psi + (\dot{x} + \mathrm{i}V)\bar{\psi}\bar{\varepsilon}). \tag{3}$$

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Hence, (2) leads to a gauge-equivalent Lagrangian and, therefore, to equations of motion identical to those obtained from the original Lagrangian (1).

The standard model of SUSY quantum mechanics [2] is found by quantizing the system (1), either within the canonical approach [3–5] or by the path-integral formalism [3, 4]. The increasing interest in this SUSY quantum model has many reasons. For recent reviews, see [6, 7]. As a particular motivation, let us mention the observation that SUSY inspired semi-classical approximations, the so-called CBC formula [8–11], in the case of unbroken SUSY and its modification [9–12] for broken SUSY, yield exact energy eigenvalues for the so-called shape-invariant potentials [13].

In contrast to SUSY quantum mechanics, which has been well studied during the last ten years, SUSY classical mechanics has, to our knowledge, never been investigated in detail†. It is the main purpose of this letter to present basic results of the classical system characterized by the Lagrangian (1). In particular, we will show that the classical solutions for the bosonic as well as the fermionic degrees of freedom are completely described by the dynamics of a real-valued *quasi-classical* degree of freedom.

The classical equations of motion, which can be derived from the Lagrangian (1), read:

$$\dot{\bar{\psi}} = \mathrm{i}V'(x)\bar{\psi} \tag{4}$$

$$\dot{\psi} = -\mathrm{i}V'(x)\psi \tag{5}$$

$$\ddot{x} = -V(x)V'(x) - V''(x)\bar{\psi}\psi \tag{6}$$

where the prime and dot denote the derivative with respect to x and t, respectively. The first-order differential equations for the fermionic degrees of freedom can immediately be integrated. With initial conditions $\psi(0) =: \psi_0$ and $\bar{\psi}(0) =: \bar{\psi}_0$, the solutions read:

$$\bar{\psi}(t) = \bar{\psi}_0 \exp\left\{\mathrm{i}\int_0^t \mathrm{d}\tau\ V'(x(\tau))\right\} \qquad \psi(t) = \psi_0 \exp\left\{-\mathrm{i}\int_0^t \mathrm{d}\tau\ V'(x(\tau))\right\} \tag{7}$$

where $x(t)$ denotes the (unknown) solution of (6). Let us note that the solutions (7) imply that $\bar{\psi}(t)\psi(t) = \bar{\psi}_0\psi_0$ is a constant and, therefore, equation (6) simplifies to

$$\ddot{x} = -V(x)V'(x) - V''(x)\bar{\psi}_0\psi_0. \tag{8}$$

As the superpotential V is assumed to be real-valued, the bosonic degree of freedom $x(t)$, which is an even Grassmann number, necessarily has the following form:

$$x(t) =: x_{\mathrm{qc}}(t) + q(t)\bar{\psi}_0\psi_0 \tag{9}$$

where $x_{\mathrm{qc}}(t)$ and $q(t)$ are real-valued functions of time. We will call $x_{\mathrm{qc}}(t)$ the *quasi-classical* solution in order to differentiate it from the full classical solution $x(t)$ which contains the $\bar{\psi}_0\psi_0$ term. The classical solution $x(t)$ and the quasi-classical solution $x_{\mathrm{qc}}(t)$ coincide only for the special initial condition $\bar{\psi}_0 = 0 = \psi_0$. It is also worth mentioning that in the fermionic solutions (7), one may replace $x(\tau)$ by $x_{\mathrm{qc}}(\tau)$ because of (9).

Multiplication of (8) by $\dot{x}$ and integration leads to the energy conservation

$$\mathcal{E} = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}V^2(x) + V'(x)\bar{\psi}_0\psi_0 \tag{10}$$

† A brief and incomplete discussion has been given in appendix C of [4].

where $\mathcal{E}$ is a constant even Grassmann number. The ansatz (9) together with $\mathcal{E} =: E + F\bar{\psi}_0\psi_0$ $(E, F \in \mathbb{R})$ results in

$$\dot{x}_{\mathrm{qc}}^2 = 2E - V^2(x_{\mathrm{qc}}) \tag{11}$$

$$\dot{q} = \frac{1}{\dot{x}_{\mathrm{qc}}}[F - V'(x_{\mathrm{qc}}) - V(x_{\mathrm{qc}})V'(x_{\mathrm{qc}})q]. \tag{12}$$

The last equation which determines $q(t)$ can also be solved exactly:

$$q(t) = \frac{\dot{x}_{\mathrm{qc}}(t)}{\dot{x}_{\mathrm{qc}}(0)}\left[q(0) + \int_0^t \mathrm{d}\tau\, \frac{F - V'(x_{\mathrm{qc}}(\tau))}{2E - V^2(x_{\mathrm{qc}}(\tau))}\right] \tag{13}$$

where $q(0)$ is a constant of integration. Again we find, as for the fermionic degrees of freedom, that $q(t)$ is expressible in terms of the quasi-classical solution $x_{\mathrm{qc}}(t)$ determined by (11). Let us note that the singularity of the integral in (13) near the turning points of the quasi-classical path is precisely cancelled by its prefactor, since $\dot{x}_{\mathrm{qc}}(t)$ vanishes at these points. Hence, $q(t)$ remains finite for all $t \geqslant 0$. Let us also note that even for the initial condition $q(0) = 0$, we have, in general, $q(t) \neq 0$ for $t > 0$. In other words, even assuming the classical solution to be initially real, $x(0) \in \mathbb{R}$ will, in general, become a Grassmann-valued quantity. It is only in the special case $V'(x) = F$, that is, for a harmonic superpotential, where a real $x(0)$ remains real for ever.

Let us now discuss some properties of the quasi-classical solution $x_{\mathrm{qc}}(t)$. The equation of motion (11) for the quasi-classical path can be obtained from a *quasi-classical* Lagrangian defined by

$$L_{\mathrm{qc}} := \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}V^2(x) = \tfrac{1}{2}(\dot{x} \pm \mathrm{i}V(x))^2 \mp \mathrm{i}V(x)\dot{x}. \tag{14}$$

The last equality shows that this Lagrangian is gauge equivalent to†

$$\tilde{L}_{\mathrm{qc}}^{\pm} := \tfrac{1}{2}(\dot{x} \pm \mathrm{i}V(x))^2. \tag{15}$$

The canonical momenta obtained from Lagrangians $\tilde{L}_{\mathrm{qc}}^{\pm}$ are

$$\xi^{\pm} := \frac{\partial \tilde{L}_{\mathrm{qc}}^{\pm}}{\partial \dot{x}} = \dot{x} \pm \mathrm{i}V(x) \tag{16}$$

and, surprisingly, coincide with the generators of the SUSY transformation (2) of the fermionic degrees of freedom

$$\delta\psi(t) = -\mathrm{i}\xi^{-}\bar{\varepsilon} \qquad \delta\bar{\psi}(t) = \mathrm{i}\xi^{+}\varepsilon. \tag{17}$$

It is also obvious that the energy E of the quasi-classical solution can be expressed by $E = \frac{1}{2}\xi^{+}\xi^{-}$. As a consequence, we have the relation

$$\xi^{\pm}/\sqrt{2E} = (\xi^{\mp}/\sqrt{2E})^{-1} \qquad E > 0. \tag{18}$$

† The reader should not be confused by the complex gauge transformation. This defect can be avoided by introducing Euclidean time.

As an aside, we mention that for $E = 0$, the quasi-classical solutions are given by $x_{qc}(t) = x_k$ where x_k are the zeros of the superpotential $V(x_k) = 0$. This leads to $\psi(t) = \psi_0$, $\bar{\psi}(t) = \bar{\psi}_0$ and $q(t) = 0$. Hence, this is the only case where a non-harmonic superpotential will lead to purely real solutions $x(t) = x_k$.

Finally, let us comment on the role played by the quasi-classical path for the quantum version of model (1). As we have already mentioned, it has been found that the SUSY-inspired semi-classical quantization does provide the exact bound-state energy spectrum for a wide class of superpotentials. Indeed, it has recently been verified that these semi-classical quantization formulae can be derived via Feynman's path-integral approach [9–11]. The important step in this derivation was to evaluate the path integral in a stationary phase approximation about these quasi-classical paths. To be more precise, instead of making the full action

$$S := \int \mathrm{d}t\, L = \int \mathrm{d}t \left[L_{qc} + \frac{\mathrm{i}}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - V'(x)\bar{\psi}\psi \right] \tag{19}$$

stationary, one calculates the corresponding path integral about the stationary paths of the quasi-classical action $S_{qc} := \int \mathrm{d}t\, L_{qc}$. This result indicates that the quasi-classical paths (and their quadratic fluctuations) carry the most important contributions of the path integral.

As a last indication for the dominance of these quasi-classical paths, let us mention an interesting result of Ezawa and Klauder [14]. These authors have shown that as long as one is only interested in expectation values of the bosonic variable $x(t)$, then the path-integral quantization based on the Lagrangian (1) is equivalent to that based on (15). To be more explicit, they showed the relation†

$$\int \left[\prod_\tau \mathrm{d}x(\tau)\, \mathrm{d}\psi(\tau)\, \mathrm{d}\bar{\psi}(\tau) \right] x(\tau_1) \cdots x(\tau_n) \exp\left\{ \left(\frac{\mathrm{i}}{\hbar}\right) \int_0^t \mathrm{d}\tau\, L \right\}$$
$$= \int \left[\prod_\tau \mathrm{d}\xi^\pm(\tau) \right] x(\tau_1) \cdots x(\tau_n) \exp\left\{ \left(\frac{\mathrm{i}}{2\hbar}\right) \int_0^t \mathrm{d}\tau\, (\xi^\pm(\tau))^2 \right\} \tag{20}$$

where $\xi^\pm(t)$ is defined by (16). It should be stressed that the last path integral is of Gaussian type. This might be the reason for the exactness of the SUSY-inspired semi-classical approximation. The gauge transformation $L_{qc} \rightarrow \tilde{L}^\pm_{qc}$ is the classical analogue of the Nicolai map discussed by Ezawa and Klauder [14].

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